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VALUES OF INFORMATION AND LIQUIDITY

PREFERENCE: A COMMENTARY NOTE

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1. Introduction

In his recent article [2] Professor Arrow analyzed the Bayesian problems of decision making under uncertainty by casting them into an information theoretic framework and proposed the concepts of the value of and demand for information. As a possible direction of extending his far-reaching ideas this note is intended to develop the following three aspects of importance in his article.

At first, we show that his definition of the value of information contains one logical slip, more precisely, a still remaining confusion of comparing the utility of income with the cost, the very same point that he keenly criticized in reference to other authors' preceding contributions in economic and statistical studies of information. In order to improve his result, we redefine the concept of the value of information in such a way that we can revive the essence of J. Marschak's proposal [10] of operationally referring the value of information as a demand price. We also obtain its precise formulation in the Arrow's special context of logarithmic utility function.

Secondly, we present a concept of the value of information in the supply sense to clarify the dual nature of the values of information viewed from its buyer's and seller's standpoints. In this regard the information



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is assigned the attribute of an object for interpersonal which flows in and out among individuals in a given economy under uncertainty.

Thirdly, we analyze the problem of a characterization of liquidity preference as behavior towards imperfect information, which seems to be intrinsic in his models of risk-bearing [1, 2]. This problem was once quantified by Marschak [9] as early as 1949 and taken up again by Radner [12] and Hirshleifer [6] rather recently, while its thorough investigation is not available yet in the current economic literature. In the context of portfolio selection theory of Markowitz [8] and Tobin [14] one is supposed to reveal his preference about a given variety of risky assets in terms of his mean-variance utility, which is a derivative from his utility function of income and probability distributions of stochastic returns of assets.¹ But in order for our analysis of liquidity to be consistent with the conventional framework of finite-state general equilibrium models under uncertainty, we do not follow this mean-variable approach. Starting directly from an individual's utility function of income in Arrow's model, we attempt to illustrate his behavior pattern towards his imperfect knowledge on an uncertain nature from the angle of his optimal liquidity holding.

Though very primitive our results are in this note, they might serve one to initiate a construction of more general models which may capture important problems in the economics of uncertainty that have been outside of the scope of traditional literatures.

2. Basic Model

Let us summarize Arrow's model [2] in the following manner. A decision maker's uncertain economic environment is assumed to be completely described

¹ Mathematical structure of this transformation from one utility to another was elaborately investigated by Richter [13] and Chipman [4].

by a finite probability space (Ω, P) and a random variable $X: \Omega \rightarrow R$ on this space, where

- (1) $\Omega = \{1, \dots, S\}$: an index set of finite S possible states of nature.
- (2) $P = \{(p_1, \dots, p_S); \sum_i p_i = 1, p_i \geq 0; i \in \Omega\}$: a decision maker's prior (objectively known or subjective) probability distribution on the occurrence of each state in Ω , where p_i is the probability that state i occurs.
- (3) $X = (X_1, \dots, X_S)$: a given structure of monetary returns from each one dollar bet on the occurrence of each state in Ω .
This amounts to saying that the decision maker who bets one dollar on state i receives X_i dollars and nothing otherwise.

The decision maker is characterized by his initially held monetary resource which is normarized to the value 1 and his von Neuman-Morgenstern utility function of income:

- (4) $U: R^+ \rightarrow R$, where $U(y)$ is assumed to be monotone increasing, differentiable and of diminishing marginal utility in income y .

His action in this economy is confined to choosing an $(S + 1)$ dimensional decision vector \tilde{a} which is restricted to the feasible set A of decisions defined as

- (5) $A = \{\tilde{a} = (a_1, \dots, a_S); \sum_i a_i + b = 1, a_i \geq 0 \text{ for all } i \in \Omega, b \geq 0\}$,
where a_i is the amount bet by him on the occurrence of state i and b is the amount retained uninvested in a liquid form out of his initial response.

¹ Each X_i can be considered as a reciprocal of unit price of each i -th security provided that Arrow regards this model as a further development of his now classic paper [1].

The decision maker will then face the problem:

- (6) Given (Ω, P) , X , U and A , maximize, with respect to a decision vector $\bar{a} \in A$, the expected value of utility;

$$EU_{\{p_i\}} = \sum_i p_i U(a_i X_i + b).$$

The situation which Arrow is concerned with is as follows. Suppose that the decision maker is presented an opportunity of acquiring a certain estimate on a true state of nature in the form of a message in a finite set given as

- (7) $\Omega^* = \{1^*, \dots, S^*\}$: an index set of S possible messages, where message j^* implies the estimate, "State j will prevail."

Since there seems to be no danger of confusion, we shall use the notation $j \equiv j^*$ as long as $j \in \Omega^*$.

More analytically speaking, the acquisition of an estimation becomes possible through a discrete communication channel describable by means of $S \times S$ matrix Q which is defined as a channel matrix¹

$$(8) \quad Q = \{q_{ji}\}; \quad \sum_j q_{ji} = 1 \text{ and } q_{ji} \geq 0 \text{ for all } i \in \Omega \text{ and } j \in \Omega^*,$$

where the i -th row and j -th column entry q_{ji} of this matrix Q signifies the conditional probability that message $j \in \Omega^*$ is sent from the channel while a true state is i .

¹ The word, "channel," does not have to be understood literally in the narrow sense of mathematical communication theory. We may regard it as an operational tool of quantitatively describing degrees of accuracy in any kind of predictive activities such as research, sampling, human verbal conversation and the like which yield certain estimation on a true state in the stochastic nature.

The unconditional and conditional probability distributions $\{p_i\}$ in (2) and $\{q_{ji}\}$ in (8) will then define the unconditional message probability distribution $\{q_j\}$ and the conditional probability distribution $\{p_{ij}\}$ as

$$(9) \quad q_j = \sum_i p_i q_{ji}; \quad \sum_j q_j = 1, \quad q_j \geq 0 \text{ for all } j \in \Omega^*$$

$$(10) \quad p_{ij} = p_i q_{ji} / q_j; \quad \sum_i p_{ij} = 1, \quad p_{ij} \geq 0 \text{ for all } i \in \Omega \text{ and } j \in \Omega^*,$$

where q_j is the probability that message j is sent from the channel and p_{ij} is the conditional probability that the true state is i when message j is sent. Given an information service in the form of a channel $Q = \{q_{ji}\}$ thus characterized, the decision maker previously ignorant on a true state up to his prior probability distribution P can now take advantage of the conditional probability distribution $\{p_{ij}\}$ to form $(S + 1)$ dimensional S decision schedule vectors $\bar{a}(j)$'s conditioned by each received message $j \in \Omega^*$, within the restriction of his feasible sets A_j 's of decision schedules defined as

$$(11) \quad A_j = \{\bar{a}_j = (a_1(j), \dots, a_S(j), b(j)); \\ \sum_i a_i(j) + b(j) = 1, \quad a_i(j) \geq 0, \quad b(j) \geq 0 \text{ for all } i \in \Omega, \\ \text{for all } j \in \Omega^*.$$

The components $a_i(j)$ and $b(j)$ in a vector \bar{a}_j signify the scheduled amount of money for bet on the occurrence of state i and amount of money retained in a liquid form as functions of a transmitted message j . The decision maker's problem (6) then assumes a new form:

(12) Given (Ω, P) , X , U , Q , and A_j 's, maximize, with respect to S decision schedule vectors $\bar{a} \in A_j$; all $j \in \Omega^*$, the expected value of the conditional utility

$$E_{\{q_j\}} (EU)_{\{p_{ij}\}} = \sum_j q_j \sum_i p_{ij} U[a_i(j)X_i + b(j)].$$

Our interest is then the evaluation of potential advantage of decision scheme (12) over the scheme (6), relative of a given additional information through the channel (8). We are also concerned with the problem of how the level of optimal liquidity changes as the configuration of information changes. In this regard we need to introduce

DEFINITION I: With respect to the solution vectors \bar{a} and \bar{a}_j 's in the problems (6) and (12)¹, we define a component b in \bar{a} as the optimal liquidity and $E_{\{q_j\}}[b(j)] \equiv \sum_j q_j b(j)$ for $b(j)$'s in \bar{a}_j 's as the optimal average liquidity.

3. Demand Value of Information

Arrow has inclined to define the arithmetic difference between the maximals (5) and (11)--which has the dimension of utility of income--as the value of information, which presumably has the dimension of money units. This approach is hardly justifiable except for the case where a utility function is linear in income. Following the fully correct approach to this problem by La Valle [7], Hirshleifer [6] and Marschak and Radner [12] we introduce

DEFINITION II: Given (Ω, P) , X , U , A , Q , A_j 's and the payment scheme of requiring to pay for information service from the final outcome of the decision maker, a real number V which satisfies the equation:

$$(13) \quad \begin{aligned} \bar{a}_j \in A_j; \quad & \max_{j \in \Omega^*} \sum_j q_j \sum_i p_{ij} U[a_i(j)X_i + b(j) - V] \\ & = \max_{\bar{a} \in A} \sum_i p_i U(a_i X_i + b) \end{aligned}$$

is defined as the demand value of information with posterior payment.

The operational significance of the value of information thus posed resides in that it is the least upper bound of the buying price of information

¹ It is plain from an elementary result of convex analysis that the problems (6) and (12) have solutions because of the assumptions we imposed on the utility function U in (4).

service, or more roughly, the maximum buying price of information in the sense that a presented information service is worth acquiring if its value V exceeds the cost C of acquiring it. This view revives the essence of Marschak's idea [10] on the reference of the value of information as a demand price, after freeing it from his dimensional problem which Arrow pointed out [2, p. 275]. This Definition II immediately leads us to

PROPOSITION I: If a utility function U in (4) is strictly increasing, continuous¹ and of diminishing marginal utility in income, then the value V of information in accordance with Definition II uniquely exists and it is non-negative.

Proof: It is given in Mathematical Appendix at the end of this note.

While the investigation in the general properties of the demand value of information remains as an important problem, our immediate concern in this note is the special case of Arrow, i.e., the case where the sure system of bets exists, i.e., the random variable X satisfies the inequality

$$(14) \quad \sum_i (1/X_i) \leq 1,$$

and where

$$(15) \quad U(y) = \log y: \text{ the base of logarithm} = \text{natural number } e.$$

Before proceeding our discussion, we have to note:

REMARK: (Arrow [2, p. 268]). Under the assumptions imposed on U in (4), if $U'(0) = +\infty$, then the decision maker will invest all his money if and only if there exists a sure system of bets expressed by (14).

It is obvious that the utility function (14) satisfies the condition in the above Remark. Confining ourselves to this Arrow's special case, we readily obtain

¹ These assumptions are slightly different from the ones given by Arrow which are written out in (4) in Section 2.

PROPOSITION II: Under the assumptions (14) and (15), the value V of information in accordance with Definition II is an exponential function of the amount of information I conveyed by the channel (8) in the sense of Shannon, more precisely,

$$(16) \quad V = (1 - e^{-I}) / \sum_i (1/X_i),$$

where

$$(17) \quad I \equiv I(Q/P) \equiv - \sum_i p_i \log p_i + \sum_j q_j \sum_i p_{ij} \log p_{ij}^1$$

Proof: Using the customary Lagrangean method of maximization, we get the optimal solutions for (6) and (12) under assumptions (14) and (15) as

$$a_i = p_i; \quad b = 0$$

$$a_i(j) = p_{ij} [1 - V \sum_k (1/X_k)] + V/X_i; \quad b(j) = 0$$

for all $i \in \Omega$ and $j \in \Omega^*$. Evaluating the right and left hand sides of the definitional equation (13) in terms of these solutions, we obtain the equation

$$- \log[1 - V \sum_k (1/X_k)] = - \sum_i p_i \log p_i + \sum_j q_j \sum_i p_{ij} \log p_{ij},$$

which amounts to the equation (16) in question.

q.e.d.

The formula (16) clarifies a rather misleading criticism of Arrow against Marschak. Having observed that the "value of information" in his own arbitrary definition turns out to be equal to the amount of information itself, Arrow concluded that if the cost of acquiring information is proportionate to the amount of information then there is no way for a decision maker to determine how much information or what channel he wants since both the value of

¹ Readers who are not familiar with elementary concepts in information theory can refer to any textbook in this area, such as Ash [3].

and cost of information are proportionate to each other [2, p. 275]. But this argument is based on his own dimensional confusion of comparing the utility of income with the cost, i.e., the same kind of comparison which Marschak made [10]. In fact, the value of information properly measured in monetary units in Proposition I is strictly concave in the amount of information so that his indeterminacy problem of the optimal amount of information does not occur at least within the conditions which he assumed.

It should be understood, however, that our criticism of Arrow using his own assumptions does not necessarily mean our full acceptance of all of his assumptions either. The proportional cost of information to its amount seems to be a very narrow assumption and there may be many economic situations in which the buyers of information face the price of information that is not quite proportional to its amount. One of the purposes of the next section is to show one such counterexample by investigating a case where the information service is a privately owned, perishable object for interpersonal exchange and which yields a supply price of information strictly convex in its amount rather than proportional to it.

4. Supply Value of Information and Other Remarks

Our investigation in the value of information in the demand sense naturally leads us to characterize the similar problem from the supply side. Let us consider a decision maker in an environment similar to the one before but who initially owns an information channel with its matrix (7) and who is ready to sell it out to somebody else. Symmetrically as in Definition II we introduce

DEFINITION III: For a similar decision maker as in Definition II who is characterized by (Ω, P) , X , U , and A and privately owns a perishable informa-

tion service¹ Q , a real number W which satisfies the equation

$$(18) \quad \max_{\bar{a}_j \in A_j; j \in \Omega^*} \sum_j q_j \sum_i p_{ij} U[a_i(j)X_i + b(j)] \\ = \max_{\bar{a} \in A} \sum_i p_i U(a_i X_i + b + W)$$

is defined as the supply value of information with posterior payment.

The value of W of information thus defined² is the greatest lower bound of the selling price of information service, or roughly, a minimum selling price information in the sense that an information service is worse selling out if the value W falls short of the revenue R . With respect to Arrow's special circumstance, we obtain

PROPOSITION III: Under the assumptions (14) and (15), the supply value W of information in accordance with Definition III is given by

$$(19) \quad W = (e^I - 1) / \sum_i (1/X_i),$$

where I is the amount of information given in (17).

The proof of this simple result may be omitted. The formula (19) illustrates that there is no universal ground to support the cost of information proportional to its amount.

So far we have been assuming the posterior payment for an information service withdrawn from or added on to the final outcomes of the decision. But if we consider the case where the monetary payment for information is taken out of or added on to the initial resource, then the sets of decision schedules given by (5) and (11) must be redefined accordingly. With respect to any real numbers \tilde{V} and \tilde{W} , let us define sets $A_{\tilde{V}}(j)$'s and $A_{\tilde{W}}$ as

¹ An information service may be said to be perishable if it does not maintain its service for owner's benefits once he sells it away.

² The proof of its unique existence and nonnegativity can be easily done similarly to the proof of Proposition I.

$$(20) \quad A_{\tilde{V}}(j) = \{\tilde{a}_j = (a_1(j), \dots, a_S(j), b(j));$$

$$\sum_i a_i(j) + b(j) = 1 - \tilde{V}, a_i(j) \geq 0 \text{ for all } i \in \Omega, b(j) \geq 0\}$$

$$(21) \quad A_{\tilde{W}} = \{\tilde{a} = (a_1, \dots, a_S, b);$$

$$\sum_i a_i + b = 1 + \tilde{W}, a_i \geq 0 \text{ for all } i \in \Omega, b \geq 0\}.$$

NOTE: In Definition II, if we restrict decision schedule vectors \tilde{a}_j 's to the sets $A_{\tilde{V}}(j)$'s instead of A_j 's, a real number \tilde{V} satisfying the equation (13) can be defined as the demand value of information with prior payment. Similarly, in Definition III, with the restriction of a decision vector \tilde{a} to $A_{\tilde{W}}$ instead of A , a real number \tilde{W} satisfying the equation (21) is defined as the supply value of information with prior payment. Under the assumptions (14) and (15), the thus defined values V and W of information are formulated as

$$(22) \quad \tilde{V} = 1 - e^{-I}$$

$$(23) \quad \tilde{W} = e^I - 1.$$

Summarizing the special results (16), (19), (22), and (23), we conclude this section with the following proposition whose proof may be unnecessary:

PROPOSITION IV: Given (Ω, P) , X , U , Q , A , A_j 's, $A_{\tilde{V}}(j)$'s, $A_{\tilde{W}}$ and assumptions (14) and (15), the demand and supply values V , \tilde{V} , W , and \tilde{W} of information with posterior and prior payments in accordance with their associated definitions are strictly increasing in the amount of information conveyed by a given channel Q relative to a given prior probability distribution P . They are nonnegative and become equal to zero if and only if the amount of information is zero, i.e., a given channel is useless.¹ Moreover, the demand

¹ A channel is said to be useless if $p_{ij} = p_i$ for all $i \in \Omega$ and $j \in \Omega^*$. For details of classification of channels, see, for example, Ash [3].

values of information are strictly concave and supply values strictly convex in the amount of information. The demand values of information with posterior and prior payments become identical functions of the amount of information if $\sum_i (1/X_i) = 1$, and the similar fact also holds for the supply values of information.

REMARK: The condition $\sum_i (1/X_i) = 1$ has a significant implication in the context of Arrow [1] if we regard $1/X_i$ as a unit price of i -th security in an uncertain pure exchange economy of C commodities with S possible states. Arrow demonstrated that the optimal allocation of $(S \times C)$ contingent commodities, which appears to require to operate $(S \times C)$ markets can be achieved by operating only $(S + C)$ markets, i.e., S for securities and C for commodities. The above condition excludes the possibility of arbitrage between securities' and commodities' markets so that this economization of markets becomes meaningful enough.

5. Liquidity Preference as Behavior Towards Imperfect Information

Our discussion in the previous two sections was so dependent on Arrow's special case conditioned by the assumptions (14) and (15), especially by (14), that the problem of liquidity preference, which his article [2] rather implicitly points out, did not actually arise in our analysis. But it should be understood that Definition I and the optimization problems (6) and (13) in Section 2 of this note have already given us the necessary framework for the analysis of optimal liquidity. In contrast to the traditional characterization of liquidity preference in terms of the mean-variance of probability distribution of risky assets, we are interested in the analysis which is directly based on a utility function of income from which the portfolio selection theorists supposedly deduce the associated mean-variance utility function.

Admittedly, the discrete description of uncertain returns from an investment may not be so practical and is quite foreign among their familiar continuous descriptions in the portfolio selection theory, except for a very few cases such as Chipman's analysis of the situation of two-point probability distribution [4, p. 181]. However, discrete models may be still interesting from a purely theoretical point of view because of their akinness to the general equilibrium models under uncertainty of Arrow-Debrue-Radner type as we mentioned in Section 1.

Generally speaking, we would like to know how a decision maker's optimal amount of liquidity holding changes as his knowledge on the uncertain nature changes due to additional information supplies to him under the condition

$$(24) \quad \sum_i (1/X_i) > 1$$

and without imposing too many assumptions on the properties of his utility function. But this general approach seems to be analytically very difficult. Therefore, we confine the scope of analysis to the case of logarithmic utility function (15) as before.

As an illustration of the nature of the problem which we are concerned with, let us consider the following simple numerical example:

$$(25) \quad \begin{aligned} \Omega &= \Omega^* = \{1, 2, 3\} \\ (p_1, p_2, p_3) &= (.05, .10, .85) \\ (X_1, X_2, X_3) &= (5, 2, 2.5)^1. \end{aligned}$$

Case 1: Given these datum and $U(\cdot) = \log(\cdot)$, the maximization problem (6) in section 2 yields the following corner optimum solution:

¹ Note that $\sum_{i=1}^3 (1/X_i) = 11/10 > 1$ so that the condition (24) is met.

$$(a_1, a_2, a_3, b) = (0, 0, .75, .25)$$

where we obtain $b = .25$ as the optimal liquidity according to Definition I.

Case 2: Let us next consider the case where the costless information service is acquired in the following form of tertiary symmetric channel¹ with error probability $\epsilon = .25$:

$$Q_{.25} = \begin{pmatrix} .75 & .125 & .125 \\ .125 & .75 & .125 \\ .125 & .125 & .75 \end{pmatrix}$$

accompanied by the message probability distribution:

$$(q_1, q_2, q_3) = (5/32, 3/16, 21/32).$$

The problem (12) yields the set of optimal solutions:

$$\begin{pmatrix} a_1(1) & a_2(1) & a_3(1) & b(1) \\ a_1(2) & a_2(2) & a_3(2) & b(2) \\ a_1(3) & a_2(3) & a_3(3) & b(3) \end{pmatrix} = \begin{pmatrix} .2 & 0 & .6 & .2 \\ 0 & 7/30 & 13/30 & 1/3 \\ 0 & 0 & 20/21 & 1/21 \end{pmatrix}.$$

The average optimal liquidity \tilde{b} in accordance with Definition I is then calculated as

$$\tilde{b} = \sum_{j=1}^3 q_j b(j) = 31/320 \approx .097 < .25 = b.$$

We notice here that the liquidity holding conditioned by the transmitted message 2 is $1/3$ and is larger than the liquidity under no information, i.e., $b = .25$ but the liquidity averaged over message probability distribution is smaller than that value .25.

Case 3: Let us observe what the average liquidity is under the more accurate tertiary symmetric channel with error probability $\epsilon = .04$:

¹ A channel characterized by a $S \times S$ matrix $Q = \|q_{ji}\|$ is called a S -ary symmetric channel with error probability ϵ if $q_{ji} = 1 - \epsilon$ for $j = i$ and $q_{ji} = \epsilon/(S-1)$ for $j \neq i$; for all $i, j = 1, \dots, S$.

$$Q_{.04} = \begin{pmatrix} .96 & .02 & .02 \\ .02 & .96 & .02 \\ .02 & .02 & .96 \end{pmatrix}$$

accompanied with the message probability distribution:

$$(q_1, q_2, q_3) = (.067, .114, .819).$$

Under this channel the set of optimal solutions can be calculated as

$$\begin{pmatrix} a_1(1) & a_2(1) & a_3(1) & b(1) \\ a_1(2) & a_2(2) & a_3(2) & b(2) \\ a_1(3) & a_2(3) & a_3(3) & b(3) \end{pmatrix} = \begin{pmatrix} 173/268 & 0 & 0 & 95/268 \\ 0 & 39/57 & 0 & 18/57 \\ 0 & 0 & 814/819 & 5/819 \end{pmatrix}.$$

The average optimal liquidity \tilde{b} is then

$$\tilde{b} = 259/4000 = .065 < .097 \approx \tilde{b}.$$

Observation of the above results in Cases 1, 2, and 3 given the initial datum (25) tells us the following facts and conjectures:

Note 1: As was clearly stated and proved in Arrow [2], the optimal liquidity holding turns out to be positive when the problem (6) or more generally the problem (12) yields corner optimum, which needs a full application of Kuhn-Tucker Theorem in a differential form for it to be solved. From a technical point of view this difficulty may be one of the reasons why a finite state approach to the liquidity preference theory based on a utility function of income of von Neuman-Morgenstern type has not developed until today.

Note 2: Even though a decision maker is assured that he will absolutely gain from investing all his money (in the above numerical example, $X_i > 1$ for $i = 1, 2, 3$), he may still prefer to hold some positive liquidity unless perfect information is given to him.

To qualify this second Note, let us first establish

LEMMA: Under the assumptions

$$(15) \quad U(\cdot) = \log(\cdot)$$

$$(24) \quad \sum_i (1/X_i) > 1$$

$$(26) \quad X_i > 1 \text{ for all } i \in \Omega,$$

the general solution to the problem (12) in Section 2 is written out as:

For all $j \in \Omega^*$ and with respect to index sets H_j and K_j which are subsets of Ω ,

$$b(j) = (1 - \sum_{h \in H_j} p_{hj}) / (1 - \sum_{h \in H_j} (1/X_h))$$

$$a_{hj} = p_{hj} - (b(j)/X_h); \text{ for } h \in H_j$$

$$a_{kj} = 0; \text{ for } k \in K_j,$$

where

$$H_j = \{i \in \Omega; a_{ij} > 0\}$$

$$K_j = \{i \in \Omega; a_{ij} = 0\} = \Omega \setminus H_j.$$

Proof: It is given in Mathematical Appendix.

From this result we immediately obtain

THEOREM: If in the above Lemma the given channel is S-ary symmetric with error probability ϵ ,¹ and if ϵ is sufficiently small, then the optimal average liquidity \tilde{b} in accordance with Definition I becomes proportional to the error probability independently of the variation of prior probability distribution $\{p_i\}$ and of system of bets $\{X_i\}$, more precisely, it is given by

¹As for the definition of symmetric channel, see the footnote of page 14 of this note.

$$\tilde{b} = \sum_{j=1}^S q_j b(j) = \frac{\epsilon}{S-1} \sum_{i=1}^S \frac{1-p_i}{1-\frac{1}{X_i}}.$$

Proof: It is also given in Mathematical Appendix.

This result captures an interesting behavior pattern of a risk averse decision maker who is characteristic in Arrow's model. Although he is absolutely sure to gain ($X_i > 1$ for all $i \in \Omega$) by investing all his money ($= 1$) on bets, he keeps a certain amount of liquidity and his liquidity preference ceases only when perfect information ($\epsilon = 0$) is given to him under the system of bets (24). In contrast to this, his liquidity holding is always equal to zero regardless of his state of knowledge and it is so even under no information if he is presented a sure system of bets ($\sum_i (1/X_i) \leq 1$). This rather drastic contrast of his behavior in two different situations may be rephrased in such a way that in the former a decision maker's knowledge on his uncertain environment does not matter at all for him to choose no liquidity as optimal while in the latter it significantly matters, and in fact, zero liquidity is chosen only accompanied by perfect knowledge on the environment.

6. Summary

For the purpose of enriching the hypothetical themes proposed in Arrow's article on the value of and demand for information and of making them operationally workable in economic models of uncertainty, we established the concept of the demand values of information as its maximum buying prices. To clarify his seeming attempt to regard information as an object of interpersonal exchange, we also defined the supply values of information as its minimum selling prices so that one can analyze the roles of information flow among individuals in an uncertain economy both from its buyer's and seller's viewpoints.

To make sure that these new value concepts are not arbitrary trivia, we proved their existence and nonnegativity under loose assumptions. With respect to Arrow's special example based on a Bernoullian logarithmic utility function of income we obtained functional forms of those values of information which exponentially increase in the amount of information in the sense of Shannon.

We also noted that his original model intrinsically contains an analytical characterization of a rational individual's liquidity preference as behavior towards imperfect information with somewhat different implications from the one in the traditional portfolio selection theory. By means of simple numerical illustrations and a limit theorem based on Kuhn-Tucker Theorem, we analyzed an interesting behavior pattern of a risk averter in his optimal liquidity holding in a sensitive or nonsensitive response to his state of knowledge on the uncertain environment.

Admittedly, most of the propositions obtained in this note have meanings only for illustrative purposes because of the assumption of logarithmic utility function, and not for a general theory. By confining our analysis within Arrow's special case, we attempted to capture a few essential problems arising in an uncertain economy, which distinguish themselves from the economic problems in a certain world and which we may easily fail to notice if we enlarge the scope of analysis too broadly.

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Mathematical Appendix

A. Proof of Proposition I in Section 3.

Let $\bar{a}_j^* = (a_1(j)^*, \dots, a_S(j)^*, b(j)^*)$; $j \in \Omega^*$ be optimal solution vectors of the problem (12) so that

$$(A-1) \quad \sum_j q_j \sum_i p_{ij} U[a_i(j)^* X_i + b(j)^*] \geq \sum_j q_j \sum_i p_{ij} U[a_i(j) X_i + b(j)]$$

for all $\bar{a}_j = (a_1(j), \dots, a_S(j), b(j))$; $j \in \Omega^*$.

And let $\bar{a}^* = (a_1^*, \dots, a_S^*, b^*)$ be an optimal solution vector of the problem (6). At first we shall prove the quantity (the expected marginal utility due to additional information)

$$(A-2) \quad \Delta EU = \sum_j q_j \sum_i p_{ij} U[a_i(j)^* X_i + b(j)^*] - \sum_i p_i U(a_i^* X_i + b^*)$$

is nonnegative. Noting the fact that $\sum_i q_{ij} = 1$ and $q_j p_{ij} = p_i q_{ji}$ for all i and j , we can rewrite (A-2) into

$$\begin{aligned} \Delta EU &= \sum_j q_j \sum_i p_{ij} U[a_i(j)^* X_i + b(j)^*] - \sum_i p_i \sum_j q_{ji} U(a_i^* X_i + b^*) \\ &= \sum_j q_j \sum_i p_{ij} U[a_i(j)^* X_i + b(j)^*] - \sum_j q_j \sum_i p_{ij} U[a_i(j)^\# X_i + b(j)^\#] \end{aligned}$$

where we artificially introduced vectors $\bar{a}_j^\#$'s defined as

$$\begin{aligned} \bar{a}_j^\# &= (a_1(j)^\#, \dots, a_S(j)^\#, b(j)^\#) \\ &= (a_1^*, \dots, a_S^*, b^*). \end{aligned}$$

Hence from the inequality (A-1), the right hand side of (A-2) must be non-negative, i.e., $\Delta EU \geq 0$.

Let us define the real-valued functions f and g in the following manner:

$$f(Z; \bar{a}_j \in A_j; j \in \Omega^*) = \sum_j q_j \sum_i p_{ij} U[a_i(j) X_i + b(j) - Z]$$

$$g(Z) = \max_{\bar{a}_j \in A_j; j \in \Omega^*} f(Z; \bar{a}_j \in A_j; j \in \Omega^*).$$

Since U is strictly increasing by assumption, $f(Z; \bar{a}_j \in A_j; j \in \Omega^*)$ is strictly decreasing in Z for each $(\bar{a}_j \in A_j; j \in \Omega^*)$. Hence $g(Z)$ is strictly decreasing

in Z . On the other hand, from the already proved fact $EU \geq 0$ we obtain

$$(A-3) \quad g(0) = \max_{\bar{a} \in A} \sum_i p_i U(a_i X_i + b).$$

If we let Z be sufficiently large, $g(Z)$ can be made not to exceed the right hand side of the equation (A-3). If we note that the continuity of U implies the continuity of g , then from the well-established property of continuous functions there exists V which satisfies

$$(A-4) \quad g(V) = \max_{\bar{a} \in A} \sum_i p_i U(a_i X_i + b)$$

and from the strict monotonicity of g we can conclude that this V is unique. If it is negative, then from the strict decreasingness of g and from (A-3) we get

$$g(V) > g(0) = \max_{\bar{a} \in A} \sum_i p_i U(a_i X_i + b).$$

Since this contradicts with (A-4) itself, V must be nonnegative.

q.e.d.

B. Proof of Lemma in Section 5.

We are going to consider the problem:

$$(B-1) \quad \text{Maximize, with respect to } \bar{a}_1 \in A_1, \dots, \bar{a}_S \in A_S,$$

$$\sum_j q_j \sum_i p_{ij} \log[a_i(j)X_i + b(j)].$$

By introducing S Lagrangean multipliers $\lambda_1, \dots, \lambda_S$, we rewrite the problem

(B-1) into the problem of maximizing

$$(B-2) \quad L(\bar{a}_1, \dots, \bar{a}_S; \lambda_1, \dots, \lambda_S) \\ = \sum_j q_j \sum_i p_{ij} \log[a_i(j)X_i + b(j)] + \sum_j \lambda_j [1 - (\sum_i a_i(j) + b(j))].$$

Conditions for maximization are:

For each $j \in \Omega^*$

$$(B-3) \quad \frac{\partial L}{\partial a_i(j)} = \frac{q_j p_{ij} X_i}{a_i(j) X_i + b(j)} - \lambda_j \leq 0; \text{ Equality holds if } a_i(j) > 0$$

$$(B-4) \quad \frac{\partial L}{\partial b(j)} = \sum_i \frac{q_j p_{ij}}{a_i(j) X_i + b(j)} - \lambda_j \leq 0; \text{ Equality holds if } b(j) > 0$$

$$(B-5) \quad \frac{\partial L}{\partial \lambda_j} = 1 - \left(\sum_i a_i(j) + b(j) \right) \leq 0; \text{ Equality holds if } \lambda_j > 0.$$

If we assume that λ_j 's are nonpositive, then this assumption violates the conditions (B-3) and (B-4) under supposedly positive values of p_i 's and X_i 's and nonnegative values of $a_i(j)$'s. Hence $\lambda_j > 0$ for all $j \in \Omega^*$ and then the conditions (B-5) must be read as

$$(B-5') \quad 1 - \left(\sum_i a_i(j) + b(j) \right) = 0 \text{ for all } j \in \Omega^*.$$

On the other hand, from one of the results of Arrow (see Remark in Section 3, page 7 of this note) we know that $b(j)$'s are all positive. Hence the conditions (B-4) must be read as

$$(B-4') \quad \sum_i \frac{p_{ij}}{a_i(j) X_i + b(j)} = \Lambda_j,$$

where we started to use the notation $\Lambda_j = \lambda_j / q_j$.

For convenience we define the index sets H_j and K_j as

$$H_j = \{i \in \Omega; a_i(j) > 0\}$$

$$K_j = \{i \in \Omega; a_i(j) = 0\}.$$

Then the conditions (B-3) yield

$$(B-6) \quad \frac{p_{hj} X_h}{a_h(j) X_h + b(j)} = \Lambda_j; \text{ for all } h \in H_j$$

$$(B-7) \quad \frac{p_{kj} X_k}{a_k(j) X_k + b(j)} = \frac{p_{kj} X_k}{b(j)} \leq \Lambda_j; \text{ for all } k \in K_j.$$

The above equation (B-6) is rewritten as

$$(B-8) \quad \frac{p_{hj}}{a_h(j)x_h + b(j)} = \Lambda_j/x_h; \text{ for all } h \in H_j.$$

Summing up both sides of (B-8) over the set H_j , we obtain

$$(B-9) \quad \sum_{h \in H} \frac{p_{hj}}{a_h(j)x_h + b(j)} = \Lambda_j \sum_{h \in H} (1/x_h).$$

Dividing the first two terms of (B-7) by x_k ($\neq 0$) and summing up the results over the set K , we obtain

$$(B-10) \quad \sum_{k \in K} \frac{p_{kj}}{a_k(j)x_k + b(j)} = \frac{1}{b(j)} \sum_{k \in K} p_{kj}.$$

Since $H \cup K = \Omega$, the equations (B-4'), (B-9) and (B-10) amount to:

$$(B-11) \quad \Lambda_j \sum_{h \in H} (1/x_h) + \frac{1}{b(j)} \sum_{k \in K} p_{kj} = \Lambda_j.$$

On the other hand, from the equations (B-6) we have

$$(B-12) \quad a_h(j) = \frac{p_{hj}}{\Lambda_j} - \frac{b(j)}{x_h} \text{ for all } h \in H.$$

Hence, by noting (B-5')

$$\begin{aligned} (B-13) \quad \sum_i a_i(j) + b(j) &= \sum_{h \in H} a_h(j) + \sum_{k \in K} a_k(j) + b(j) \\ &= (1/\Lambda_j) \sum_{h \in H} p_{hj} - b(j) \sum_{h \in H} (1/x_h) + b(j) \\ &= 1. \end{aligned}$$

From the equation (B-11) we know

$$(B-14) \quad b(j) \sum_{h \in H} (1/x_h) = b(j) - (1/\Lambda_j) \sum_{k \in K} p_{kj}.$$

From (B-13) and (B-14) we obtain

$$(1/\Lambda_j) \sum_i p_{ij} = 1,$$

which amount to $\Lambda_j = 1$ (i.e., $\lambda_j = q_j$) for all $j \in \Omega^*$ since $\sum_i p_{ij} = 1$ for all $j \in \Omega^*$. Therefore, (B-14) yields

$$(B-15) \quad b(j) = \frac{\sum_{k \in K} p_{kj}}{1 - \sum_{h \in H} (1/X_h)} = \frac{1 - \sum_{h \in H} p_{hj}}{1 - \sum_{h \in H} (1/X_h)} ; j \in \Omega^*.$$

In terms of these solutions for $b(j)$'s we obtain

$$a_i(j) = \begin{cases} p_{ij} - b(j)/X_i & \text{if } i \in H_j \\ 0 & \text{if } i \in K_j \end{cases}$$

q.e.d.

C. Proof of Theorem in Section 5.

If a given channel $Q = ||q_{ji}||$ is S-ary symmetric with error probability ϵ , i.e., $q_{ji} = 1 - \epsilon$ for $j = i$ and $q_{ji} = \epsilon/(S-1)$ for $j \neq i$, then $p_{ij} \rightarrow 1$ if $i = j$ and $p_{ij} \rightarrow 0$ if $i \neq j$ as $\epsilon \rightarrow 0$ and the sets H_j 's in the above proof of Lemma shrink to singleton sets $\{j\}$'s. Hence from the result (B-15) we get

$$(C-1) \quad b(j) = (1 - p_{jj})/(1 - (1/X_j)); \text{ for all } j \in \Omega^*.$$

We further obtain

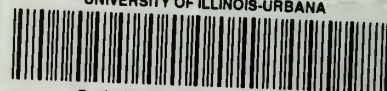
$$\begin{aligned} q_j(1 - p_{jj}) &= q_j - q_j p_{jj} = \sum_i p_i q_{ji} - p_j q_{jj} \\ &= \sum_{i \neq j} p_i q_{ji} = \frac{\epsilon}{S-1} \sum_{i \neq j} p_i \\ &= \frac{\epsilon}{S-1} (1 - p_j) \end{aligned}$$

Combining these results with (C-1), we obtain the optimal average liquidity \tilde{b} as

$$\tilde{b} \equiv \sum_{j \in \Omega^*} q_j b(j) = \frac{\epsilon}{S-1} \sum_{j \in \Omega^*} \frac{1 - p_j}{1 - \frac{1}{X_j}}.$$

q.e.d.

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